MATEMATIKA 3: Ispit se održava sukladno objavljenim pravilima. Na snazi je Pravilnik o stegovnoj odgovornosti studenata.

IME I PREZIME: RJEŠENJE 1 BROJ INDEKSA:

Grupa xxoxx
POPUNJAVA NASTAVNIK
Broj ↓ bodova

1. Koristeći Laplaceovu transformaciju riješiti diferencijalnu jednadžbu: $2x'''(t) + 5x'(t) = t$, $x(0) = 1$ i $x'(0) = x''(0) = 0$. 20

2. Neka je K kugla radijusa $r = 1$ sa centrom u ishodistu. Preko definicije plošnog integrala izračunati $\iint\limits_{\partial K} 3\,\mathrm{d}S$. 20

3. Neka je K kugla radijusa $r = 2$ sa centrom u ishodistu. Izraèunati $\iiint\limits_{K} (2x+3)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$. 20

4. Neka je K krug radijusa $r = 2$ sa centrom u toèki $T(0,0)$. Izraèunati $\int\limits_{\partial K} (2x+3)\,\mathrm{d}s$. 20

5. Neka je $\widehat{\Gamma} = \left\{(x,y,z) : x = \frac{1}{2}\cos t,\ y = \frac{1}{2}\sin t,\ z = \frac{\sqrt{3}}{2},\ t \in [0,\pi]\right\}$ i $\mathbf{w}(x,y,z) = (y,z,x)$. Izraèunati $\int\limits_{\widehat{\Gamma}} (\mathbf{w}|\mathrm{d}\mathbf{r})$. 20

Ukupno:

Tablica integrala

$\int \mathrm{d}x = x + C$	$\int \frac{\mathrm{d}x}{\cos^2 x} = \tan x + C$	$\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C$
$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1},\ n \neq -1$	$\int \frac{\mathrm{d}x}{\sin^2 x} = -\cot x + C$	$\int \frac{\mathrm{d}x}{a^2-x^2} = \frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| + C$
$\int \frac{\mathrm{d}x}{x} = \ln\|x\| + C$	$\int \sinh x\,\mathrm{d}x = \cosh x + C$	$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| + C$
$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + C$	$\int \cosh x\,\mathrm{d}x = \sinh x + C$	$\int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln\left\|x + \sqrt{x^2 \pm a^2}\right\| + C$
$\int \sin x\,\mathrm{d}x = -\cos x + C$	$\int \tanh x\,\mathrm{d}x = \ln\|\cosh x\|$	$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$
$\int \cos x\,\mathrm{d}x = \sin x + C$	$\int \coth x\,\mathrm{d}x = \ln\|\sinh x\|$	$\int \frac{\mathrm{d}x}{\sqrt{2ax-x^2}} = \arccos\left(1-\frac{x}{a}\right) + C$
$\int \tan x\,\mathrm{d}x = -\ln\|\cos x\|$	$\int \frac{\mathrm{d}x}{\cosh^2 x} = \tanh x + C$	$\int \sqrt{x^2 \pm a^2}\,\mathrm{d}x = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x + \sqrt{x^2 \pm a^2}\right)\right]$
$\int \cot x\,\mathrm{d}x = \ln\|\sin x\|$	$\int \frac{\mathrm{d}x}{\sinh^2 x} = -\coth x + C$	$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2\arcsin\left(\frac{x}{a}\right)\right] + C$

① $2x'''(t)+5x'(t)=t \quad , \quad x(0)=1 \; , \; x'(0)=0 \; , \; x''(0)=0$

$$2\left(s^3X(s) - s^2\underbrace{X(0)}_{=1} - s\underbrace{\cancel{X'(0)}}_{0} - \underbrace{\cancel{X''(0)}}_{0}\right) + 5\left(sX(s) - \underbrace{X(0)}_{=1}\right) = \frac{1}{s^2}$$

$$X(s) = \frac{\frac{1}{s^2}+5+2s^2}{s(2s^2+5)} = \frac{2s^4+5s^2+1}{s^3(2s^2+5)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{Ds+E}{2s^2+5}$$

$$2s^4+5s^2+1 = (As^2+Bs+C)(2s^2+5)+(Ds+E)s^3$$

POSTAVITI $s=0 \Rightarrow 1 = 5C \Rightarrow C=\frac{1}{5}$ I UVRSTI U GORE

$$2s^4+5s^2+1-\frac{1}{5}(2s^2+5) = (As^2+Bs)(2s^2+5)+(Ds+E)s^3$$

$$2s^4+5s^2+\cancel{1}-\frac{2}{5}s^2-\cancel{1} = \dots$$

$$s^2\left(2s^2+\frac{23}{5}\right) = \cancel{s}\left[(As+B)(2s^2+5)+(Ds+E)s^2\right]$$

POSTAVITI $s=0 \Rightarrow 0 = 5B \Rightarrow B=0$ I UVRSTI GORE PA SLIJEDI:

$$s\left(2s^2+\frac{23}{5}\right) = As(2s^2+5)+(Ds+E)s^2$$

$$\cancel{s}\left(2s^2+\frac{23}{5}\right) = \cancel{s}\left[A(2s^2+5)+(Ds+E)s\right]$$

POSTAVITI $s=0 \Rightarrow +\frac{23}{5} = 5A \Rightarrow A=+\frac{23}{25}$ UVRSTI GORE, SLIJEDI:

$$2s^2+\frac{23}{5} = +\frac{23}{25}(2s^2+5)+(Ds+E)s$$

$$2s^2+\cancel{\frac{23}{5}} = +\frac{46}{25}s^2+\cancel{\frac{23}{5}}+Ds^2+Es$$

UZ s^2: $2 = +\frac{46}{25}+D \Rightarrow D=\frac{4}{25}$

UZ s: $0 = E$

$$\boxed{\frac{s}{2s^2+5} = \frac{1}{2}\frac{s}{s^2+\frac{5}{2}}}$$

$$\Rightarrow X(s) = +\frac{23}{25}\cdot\frac{1}{s}+\frac{1}{5}\cdot\frac{1}{s^3}+\frac{4}{25}\cdot\frac{s}{2s^2+5}$$

RJEŠENJE:

$$x(t) = +\frac{23}{25}+\frac{1}{10}t^2+\frac{2}{25}\cdot\frac{1}{\cancel{2}}\cdot\cos\left(\sqrt{\frac{5}{2}}\,t\right)$$

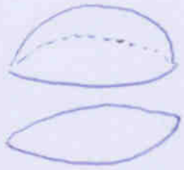
$$x(t) = \frac{1}{50}\left(46+5t^2+4\cos\left(\sqrt{\frac{5}{2}}\,t\right)\right)$$

(2) K kugla sa $r=1$, centar u $T(0,0,0)$. TRAŽI SE $\iint_{\partial K} 3\,dS$

$$\iint_{\partial K} 3\,dS = \iint_{D} 3\,\|\vec{n}(u,v)\|\,du\,dv \quad \text{gdje je } r: D \to \partial K$$

parametrizacija plohe

Parametrizacija gornje polovice sfere



sfera ∂K $\uparrow r$ krug D

2 puta po polovica sfere čini cijelu sferu.

D je krug radijusa $r=1$, koordinate su (u,v)

$$r(u,v) = (u, v, \sqrt{1-u^2-v^2}) \qquad \frac{\partial r}{\partial u} = \begin{pmatrix} 1 \\ 0 \\ \frac{-u}{\sqrt{1-u^2-v^2}} \end{pmatrix} \quad \frac{\partial r}{\partial v} = \begin{pmatrix} 0 \\ 1 \\ \frac{-v}{\sqrt{1-u^2-v^2}} \end{pmatrix}$$

$$\|\vec{n}(u,v)\| = \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| = \left\| \begin{pmatrix} \frac{u}{\sqrt{1-u^2-v^2}} \\ \frac{+v}{\sqrt{1-u^2-v^2}} \\ 1 \end{pmatrix} \right\| = \sqrt{\frac{u^2+v^2}{1-u^2-v^2} + 1}$$

$$= \sqrt{\frac{u^2+v^2+1-u^2-v^2}{1-u^2-v^2}} = \frac{1}{\sqrt{1-u^2-v^2}}$$

$$\iint_{\partial K} 3\,dS = 2\iint_{D} 3\,\|\vec{n}(u,v)\|\,du\,dv = 6\iint_{D} \frac{du\,dv}{\sqrt{1-u^2-v^2}} = 6\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr\,d\varphi$$

polarne koordinate
$u = r\cos t$
$v = r\sin t$

$$= 6 \cdot 2\pi \cdot \int_0^1 \frac{r}{\sqrt{1-r^2}}\,dr$$

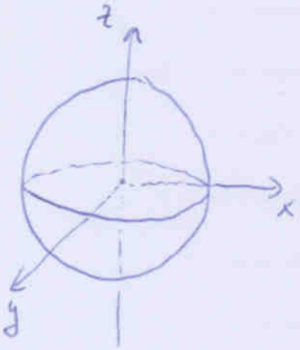
$$= 6 \cdot 2\pi \cdot \left(-\sqrt{1-r^2}\right)\Big|_0^1$$

$$= 6 \cdot 2\pi \cdot (0+1) = 12\pi$$

③

$$\iiint_K (2x+3)\,dx\,dy\,dz = \int_0^{2\pi}\int_{-2}^{2}\int_0^{\sqrt{1-z^2}} (2r\cos\varphi+3)\,r\,dr\,dz\,d\varphi$$

$x = r\cos\varphi$
$y = r\sin\varphi$
$z = z$



$\varphi \in [0, 2\pi]$
$z \in [-2, 2]$
$r \in [0, \sqrt{4-z^2}]$

$$= 2\pi \int_{-2}^{2}\int_0^{\sqrt{4-z^2}} 3r\,dr\,dz$$

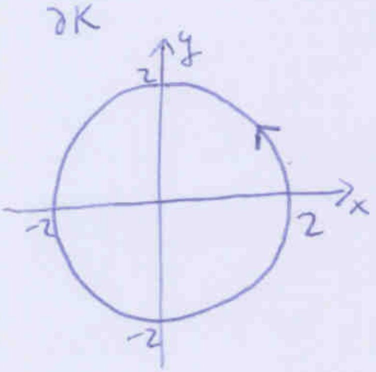
$$= \frac{6\pi}{2}\int_{-2}^{2} 4-z^2\,dz$$

$$= 3\pi\left(4z - \frac{z^3}{3}\right)\Big|_{z=-2}^{z=2}$$

$$= 3\pi\left[4\cdot 4 - \frac{1}{3}(8+8)\right] = 32\pi$$

④ Neka je K krug radijusa $r=2$ s centrom $T(0,0)$.

$$\iint_{\partial K} (2x+3)\,ds = \left\{\begin{matrix}\text{krivuljni}\\ \text{integral}\\ \text{1. vrste}\end{matrix}\right\} = \int_0^{2\pi} (2\cdot 2\cos t + 3)\cdot 2\,dt = 12\pi$$



parametrizacija kružnice

$r(t) = (2\cos t, 2\sin t) \Rightarrow r'(t) = (-2\sin t, 2\cos t)$
$t \in [0, 2\pi]$
$\|r'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t} = 2$

⑤ Parametrizacija krivulje $\tilde{\Gamma} = \{(x,y,z) \mid x = \frac{1}{2}\cos t,\ y = \frac{1}{2}\sin t,\ z = \frac{\sqrt{3}}{2},\ t \in [0,\pi]\}$

$\Rightarrow r(t) = \left(\frac{1}{2}\cos t, \frac{1}{2}\sin t, \frac{\sqrt{3}}{2}\right);\ r'(t) = \left(-\frac{1}{2}\sin t, \frac{1}{2}\cos t, 0\right)$

$w(x,y,z) = (y,z,x) \Rightarrow w(r(t)) = \begin{pmatrix}\frac{1}{2}\sin t\\ \frac{\sqrt{3}}{2}\\ \frac{1}{2}\cos t\end{pmatrix}$

$w(r(t))\cdot r'(t) = -\frac{1}{4}\sin^2 t + \frac{\sqrt{3}}{4}\cos t$

skalarno množenje

$$\int_{\tilde{\Gamma}} w\cdot dr = \int_0^{\pi} -\frac{1}{4}\sin^2 t + \frac{\sqrt{3}}{4}\cos t\,dt$$

$$= -\frac{1}{4}\int_0^{\pi}\sin^2 t\,dt + \frac{\sqrt{3}}{4}\underbrace{\int_0^{\pi}\cos t\,dt}_{=0} = -\frac{1}{4}\int_0^{\pi}\frac{1-\cos 2t}{2} = -\frac{\pi}{8}$$