MATEMATIKA 2

14. lipnja 2012.

Ime i prezime: RJEŠENJE 3 **Broj indeksa:**

Vrijeme: od ____________ **do** _________ ♣3 **Broj bodova:**

Trajanje ispita je 120 minuta. Ispit se održava sukladno objavljenim pravilima. Na snazi je Pravilnik o stegovnoj odgovornosti studenata.

1. *(15)* Integriraj

$$\int_1^2 \frac{\sin(\ln x)}{x} dx$$

2. *(20)* Integriraj

$$\int \frac{x}{(x-1)(x^2+x+1)} dx$$

3. *(20)* Odredi površinu koju zatvaraju krivulja $x = y^2 - 2y + 2$ i pravac $2x + y = 9$.

4. *(10+10)*

a) Ispitaj ekstreme funkcije

$$f(x,y) = x^3 + y^3 - 15xy$$

b) Odredi domenu funkcije:

$$f(x,y) = \arcsin(x+y)$$

5. *(10+15)* Riješi sljedeće diferencijalne jednadžbe:

a)

$$xy' - 4y = x^3$$

b)

$$y'' + 9y = 2e^{-3x}$$

5b) HOMOGENA

$y'' + 9y = 0$

KARAKTERISTIČNA

$\lambda^2 + 9 = 0$

$\lambda_{1,2} = \pm 3i$

$= 0 \pm 3i$

OPĆE RJEŠ. HOMOGENE

$y_0 = A\cos(3x) + B\sin(3x)$

PARTIKULARNO RJEŠENJE POLAZNE JEDNADŽBU U OBLIKU:

$Y = Ce^{-3x} \Rightarrow Y' = -3Ce^{-3x}$, $Y'' = 9Ce^{-3x}$

UVRSTITI U $Y'' + 9Y = 2e^{-3x}$

$9Ce^{-3x} + 9Ce^{-3x} = 2e^{-3x}$

$18Ce^{-3x} = 2e^{-3x}$

$18C = 2$

$C = \frac{1}{9}$

$\Rightarrow Y = \frac{1}{9}e^{-3x}$

OPĆE RJEŠENJE NEHOMOGENE POLAZNE ODJ.

$y(x) = y_0 + Y = A\cos(3x) + B\sin(3x) + \frac{1}{9}e^{-3x}$

PISATI JEDNOSTRANO!

NA SVAKI LIST PAPIRA NAPISATI IME I PREZIME

1) $$\int \frac{\sin(\ln x)}{x}\,dx = \int \sin(\ln x)\cdot\frac{dx}{x} = \left\{\begin{matrix} u=\ln x \\ du=\frac{1}{x}dx \end{matrix}\right\}$$

$$= \int \sin u\, du = -\cos u = -\cos(\ln x) + C$$

$\underline{\mathcal{D}(\text{primitivne f-je}) = \mathbb{R}\setminus\{0\},\ [1,2] \subseteq \mathcal{D}(\uparrow)}$

$$\int_1^2 \frac{\sin(\ln x)}{x}\,dx = \Big[-\cos(\ln x)\Big]_1^2$$

$$= -\underbrace{\cos(\ln 2)}_{\substack{\text{KALKULATOR}\\ \approx 0.77}} + \underbrace{\cos(\underbrace{\ln 1}_{=0})}_{=1} \approx 0.23$$

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2) $$\int \frac{x}{(x-1)(x^2+x+1)}\,dx = \int \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\,dx$$

$$\Rightarrow \quad x = A(x^2+x+1) + (Bx+C)(x-1)$$

$$\Rightarrow \begin{cases} A = \frac{1}{3} \\ B = -\frac{1}{3} \\ C = \frac{1}{3} \end{cases}$$

$$= \underbrace{\frac{1}{3}\int\frac{dx}{x-1}}_{=\frac{1}{3}\ln|x-1|} + \frac{1}{3}\int\frac{-x+1}{x^2+x+1}\,dx = (*)$$

$$\int \frac{-x+1}{x^2+x+1}\,dx = \left\{\begin{matrix} \text{TIP } B \neq C; \\ t = x^2+x+1 \\ dt = 2x+1 \end{matrix}\right\} = \underbrace{\int \frac{-\frac{1}{2}(2x+1)}{x^2+x+1}\,dx}_{=-\frac{1}{2}\ln|x^2+x+1|} + \underbrace{\int\frac{+\frac{1}{2}+1}{x^2+x+1}\,dx}_{=\frac{3}{2}\int\frac{dx}{x^2+x+1}}$$

$$\int\frac{dx}{x^2+x+1} = \int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \left\{\begin{matrix} t = x+\frac{1}{2} \\ dt = dx \end{matrix}\right\} = \int\frac{dt}{t^2+\frac{3}{4}} = \frac{1}{\sqrt{\frac{3}{4}}}\arctan\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}} + C$$

$$(*) = \frac{1}{3}\ln|x-1| + \frac{1}{3}\cdot\frac{-1}{2}\ln|x^2+x+1| + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}} + C$$

4a) $f(x,y) = x^3 + y^3 - 15xy$

$\partial_x f = 3x^2 - 15y$

$\partial_y f = 3y^2 - 15x$

IZJEDNAČIMO S NULOM

$3x^2 - 15y = 0 \Rightarrow y = \frac{x^2}{5}$

$3y^2 - 15x = 0$

$3 \cdot \frac{x^4}{25} - 15x = 0 \quad / \cdot \frac{25}{3}$

$x^4 - 125x = 0$

$x(x^3 - 125) = 0$

$x = 0$ i $\underbrace{x^3 = 125}_{\Rightarrow x = 5}$

$\Downarrow \qquad \Downarrow$

$y = 0 \qquad y = 5$

2 KRITIČNE TOČKE

$T_1(0,0)$

$T_2(5,5)$

DALJE RAČUNATI PARC. DER. 2. REDA

$\partial_{xx} f = 6x$

$\partial_{yx} f = -15$

$\partial_{xy} f = -15$

$\partial_{yy} f = 6y$

ZA $T_1(0,0)$ MATRICA KOJU PROMATRAMO JE:

$$\begin{bmatrix} 0 & -15 \\ -15 & 0 \end{bmatrix} \Rightarrow \det = 0 \cdot 0 - (-15)(-15) = -225 < 0$$

$\Rightarrow T_1(0,0)$ NIJE LOK. EKSTREM

ZA $T_2(5,5)$ MATRICA KOJU PROMATRAMO JE:

$$\begin{bmatrix} 30 & -15 \\ -15 & 30 \end{bmatrix} \Rightarrow \det = 900 - 225 = 675 > 0$$

$\Rightarrow T_2(5,5)$ JE LOKALNI EKSTREM

ZBOG $30 > 0$ $T_2(5,5)$ JE TOČKA LOKALNOG MINIMUMA

TAJ LOKALNI MINIMUM JE VRIJEDNOST $f(5,5) = 5^3 + 5^3 - 15 \cdot 5 \cdot 5 = -125$

3) KRIVULJE: $\begin{cases} x = y^2 - 2y + 2 \\ 2x + y = 9 \end{cases}$

KADA ZAMIJENIMO X-Y OSI POSTAJU:

$$\begin{cases} y = x^2 - 2x + 2 \\ 2y + x = 9 \end{cases} \Rightarrow \begin{cases} y = x^2 - 2x + 2 \\ y = -\frac{1}{2}x + \frac{9}{2} \end{cases}$$

POVRŠINE KOJE ZATVARAJU KRIVULJE OSTAJU ISTE

SJECIŠTA DVIJE KRIVULJE:

$$x^2 - 2x + 2 = -\frac{1}{2}x + \frac{9}{2}$$

$$x^2 - \frac{3}{2}x - \frac{5}{2} = 0$$

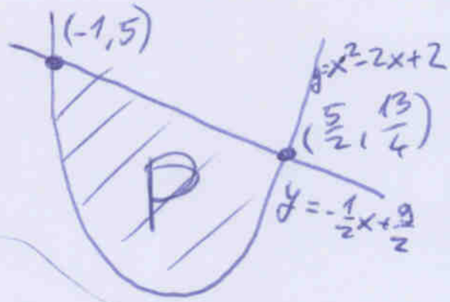
$$2x^2 - 3x - 5 = 0 \rightarrow x_{1,2} = \frac{3 \pm \sqrt{9 + 4 \cdot 2 \cdot 5}}{2 \cdot 2} = \frac{3 \pm 7}{4}$$

$$x_1 = \frac{5}{2} \quad x_2 = -1$$

$$y_1 = \frac{13}{4} \quad y_2 = 5$$

$$P = \int_{-1}^{\frac{5}{2}} \left(x^2 - 2x + 2\right) - \left(-\frac{1}{2}x + \frac{9}{2}\right) dx$$

SLIKA:



$$= \int_{-1}^{5/2} x^2 - \frac{3}{2}x - \frac{5}{2}\, dx$$

$$= \left[\frac{x^3}{3} - \frac{3x^2}{4} - \frac{5}{2}x\right]_{-1}^{\frac{5}{2}}$$

$$= \frac{1}{3} \cdot \frac{125}{8} - \frac{3}{4} \cdot \frac{25}{4} - \frac{5}{2} \cdot \frac{5}{2} - \frac{1}{3}(-1)^3 + \frac{3}{4} \cdot (-1)^2 + \frac{5}{2}(-1)$$

$$\approx -7.1458\dot{3}$$

POVRŠINA NE MOŽE BITI NEGATIVNA

AH GREŠKA U PREDZNAKU ZBOG GREŠKE OVDJE TREBA GORE ZAMIJENITI PRIBROJNIKE

$$P = \int_{-1}^{\frac{5}{2}} \left(-\frac{1}{2}x + \frac{9}{2}\right) - \left(x^2 - 2x + 2\right) dx = \ldots = 7.1458\dot{3}$$

5a) $xy' - 4y = x^3$ OVO JE LINEARNA ODJ 1. REDA

HOMOGENI DIO

$xy' - 4y = 0$

RJEŠENJE HOMOGENOG DIJELA

$xy' = 4y$

$$\frac{y'}{y} = \frac{4}{x} \Big/ \int \Rightarrow \ln|y| = 4\ln|x| \overset{+C}{=} \ln|x|^4 + C$$

$$\Rightarrow \quad y = \pm C x^4$$

RJEŠENJE POLAZNE NEHOMOGENE ODJ 1. REDA VARIJACIJOM KONSTANTE

$$y(x) = C(x) \cdot x^4$$

UVRSTIMO U POLAZNU ODJ

$$x \cdot (C(x) \cdot x^4)' - 4 \cdot (C(x) \cdot x^4) = x^3$$

$$x^5 \cdot C'(x) \cancel{+ 4x^4 \cdot C(x)} \cancel{- 4x^4 C(x)} = x^3$$

$$x^5 \cdot C'(x) = x^3$$

$$C'(x) = \frac{x^3}{x^5} = x^{-2}$$

$$C(x) = \int x^{-2} = -\frac{1}{x} + C$$

RJEŠENJE POLAZNE JEDNADŽBE

$$\Rightarrow \quad y(x) = \left(-\frac{1}{x} + C\right) \cdot x^4$$

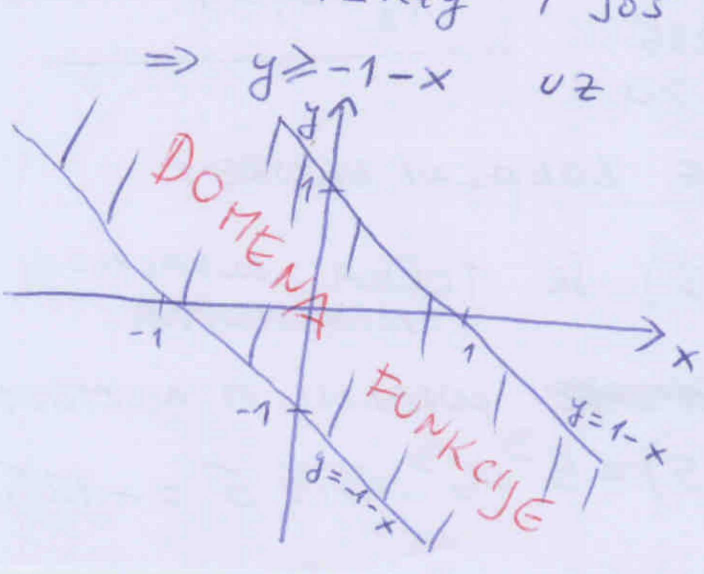
4b) $f(x,y) = \arcsin(x+y)$ $\quad \mathcal{D}(\arcsin) = [-1, 1]$

$\Rightarrow x+y \in [-1, 1]$

$\Rightarrow -1 \leq x+y \leq 1$

MORA BITI $-1 \leq x+y$ I JOŠ $x+y \leq 1$

$\Rightarrow y \geq -1 - x$ UZ $y \leq 1 - x$



DOMENA f-JE JE SKUP TOČAKA KOORDINATNE RAVNINE OZNAČEN NA SLICI: PRUGA KOJA UKLJUČUJE PRAVCE NA RUBU